A medians and radii inequality **June 2013**

1922. *Proposed by Arkady Alt, San Jose, CA.*

Let m_a, m_b, and m_c be the lengths of the medians of a triangle with circumradius R and inradius r. Prove that

$$m_a m_b + m_b m_c + m_c m_a \le 5R^2 + 2Rr + 3r^2.$$

Solution by George Apostolopoulus, Messolonghi, Greece.

Let E, F, and G be the midpoints of the sides AB, BC, and CA, respectively. We apply Ptolemy's Inequality to quadrilaterals $AEBD$, $BFEC$, and $CAFD$ to obtain

$$m_a m_b \le \frac{ab}{4} + \frac{c^2}{2}, \quad m_b m_c \le \frac{bc}{4} + \frac{a^2}{2}, \quad \text{and} \quad m_c m_a \le \frac{ca}{4} + \frac{b^2}{2}.$$

Adding up these inequalities, we get

$$m_a m_b + m_b m_c + m_c m_a \le \frac{a^2 + b^2 + c^2}{2} + \frac{ab + bc + ca}{4}.$$

By Heron's formula and $4srR = abc$ (equivalent to area $(ABC) = rs = \frac{1}{2}bc \sin A = abc/(4R)$), we get

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4rR) \quad \text{and} \quad ab + bc + ca = s^2 + r^2 + 4rR,$$

which means that

$$m_a m_b + m_b m_c + m_c m_a \le \frac{1}{4}(5s^2 - 3r^2 - 12rR).$$

Finally, by Gerretsen's Inequality $s^2 \le 4R^2 + 4rR + 3r^2$, we obtain the result

$$\begin{aligned} m_a m_b + m_b m_c + m_c m_a &\le \frac{5(4R^2 + 4rR + 3r^2) - 3r^2 - 12rR}{4} \\ &= 5R^2 + 2Rr + 3r^2. \end{aligned}$$

Equality holds when the triangle ABC is equilateral.

Also solved by Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Erhard Braune (Austria), Marian Dincă (Romania), Omran Kouba (Syria), Elias Lampakis (Greece), Moti Levy, (Israel), Peter Nüesch (Switzerland), Paolo Perfetti (Italy), and the proposer.